A further simplification can be had for PBIB and circulant designs, where $\sum_j c_{ij}^2$ is the same for all i.

For the other three criteria, elegant expressions are seldom available. Since E_4 follows directly from the **C** matrix it is easiest to compute; we do not have to solve the normal equations or evaluate the λ's.

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REFERENCES

[1] SYLVAIN EHRENFELD, "On the efficiency of experimental designs," *Ann. Math. Stat.*, Vol. 26 (1955), pp. 247–255.
[2[OSCAR KEMPTHORNE, "The efficiency factor of an incomplete block design," *Ann. Math. Stat.*, Vol. 27 (1956), pp. 846–849.
[3] J. KIEFER, "On the non-randomised optimality and randomised non-optimality of symmetrix design," *Ann. Math. Stat.*, Vol. 29 (1958), pp. 675–699.
[4] A. M. KSHIRSAGAR, "A note on incomplete block designs," *Ann. Math. Stat.*, Vol. 29 (1958), pp. 907–910.
[5] H. K. NANDI, "On the efficiency of experimental designs," *Calcutta Stat. Assoc. Bull.*, Vol. 3 (1950), pp. 167–171.
[6] ABRAHAM WALD, "On the efficient design of statistical investigations," *Ann. Math. Stat.*, Vol. 14 (1943), pp. 134-140.

ON THE COMPLETENESS OF ORDER STATISTICS[1]

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1. Introduction and summary. Let $X_1, X_2, \cdots, X_n$ be a sample of a one-dimensional random variable X; let the order statistic $T(X_1, X_2, \cdots, X_n)$ be defined in such a manner that $T(x_1, x_2, \cdots, x_n) = (x^{(1)}, x^{(2)}, \cdots, x^{(n)})$ where $x^{(1)} \leqq x^{(2)} \leqq \cdots \leqq x^{(n)}$ denote the ordered x's; and let Ω be a class of one-dimensional *cpf*'s, i.e., cumulative probability functions.

The order statistic, T, is said to be a complete statistic with respect to the class, $\{P^{(n)} \mid P \varepsilon \Omega\}$, of n-fold power probability distributions if

$$E_{P^{(n)}}\{h[T(X_1, \cdots, X_n)]\} = 0$$

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for all $P \varepsilon \Omega$ implies $h[T(x_1, \cdots, x_n)] = 0$, a.e., $P^{(n)}$, for all $F \epsilon \Omega$. The class Ω is said to be symmetrically complete whenever the latter condition holds.

Since the completeness of the order statistic plays an essential role in nonparametric estimation and hypothesis testing, e.g., Fraser [2] and Bell [1], it is of interest to determine those classes of *cpf*'s for which the order statistic is complete.

Many of the traditionally studied classes of *cpf*'s on the real line are known to be symmetrically complete, e.g., all continuous *cpf*'s ([4], pp. 131–134, 152–153); all *cpf*'s absolutely continuous with respect to Lebesgue measure ([3], pp. 23–31); and all exponentials of a certain form ([4], pp. 131–134).

The object of this note is to present a different ([4], pp. 131–134, 152–153) demonstration of the symmetric completeness of the class of all continuous *cpf*'s; and to extend this and other known completeness results to probability spaces other than the real line, e.g., Fraser [2], and Lehmann and Scheffé [5], [6].

The paper is divided into four sections. Section 1 contains the introduction and summary. In Section 2 the notation and terminology are introduced. The main theorem is presented in Section 3, and some consequences of the proof of the main theorem and known results are indicated in Section 4.

2. Terminology and notation. Let $(X, \mathcal{S})$ be an arbitrary measurable space; λ, an arbitrary measure on $(X, \mathcal{S})$; and Ω, a class of probability measures on $(X, \mathcal{S})$.

Consistent with the notation of Scheffé [7] one defines the following sets and classes.

$\Omega_0(X)$ = the class of all probability measures on $(X, \mathcal{S})$;

$\Omega_1(X)$ = the class of all nondegenerate probability measures on $(X, \mathcal{S})$;

$\Omega_2(X)$ = the class of all nonatomic probability measures on $(X, \mathcal{S})$;

$\Omega_3(\lambda) = \{P \varepsilon \Omega_0(X) \mid P \ll \lambda\}$, i.e., the class of probability measures absolutely continuous with respect to λ;

$\Omega(\mathfrak{J}, \lambda) = \{\lambda_A \mid A \varepsilon \mathfrak{J}_\lambda^+\}$ where $\mathfrak{J}_\lambda^+ = \{A \varepsilon \mathfrak{J} \mid 0 < \lambda(A) < \infty\}$ and $\lambda_A(C) = \lambda(AC)/\lambda(A)$ for all $C \varepsilon \mathcal{S}$;

$\mathfrak{N}_\Omega = \{A \varepsilon \mathcal{S} \mid P(A) = 0 \text{ for all } P \varepsilon \Omega\}$, i.e., the null class of Ω;

$(X^{(n)}, \mathcal{S}^{(n)})$ = the product n-space generated by $(X, \mathcal{S})$;

$\lambda^{(n)} = \lambda \times \cdots \times \lambda$ = the n-fold power measure on $(X^{(n)}, \mathcal{S}^{(n)})$ generated by λ;

$\Omega^{(n)} = \{P^{(n)} \mid P \varepsilon \Omega\}$ = class of power measures generated by Ω;

$\mathfrak{N}_{\Omega^{(n)}} = \{A \varepsilon \mathcal{S}^{(n)} \mid P^{(n)}(A) = 0 \text{ for all } P \varepsilon \Omega\}$ = null class of $\Omega^{(n)}$.

A class Ω is said to be *symmetrically complete for* $n = k$ if $h_k \equiv 0[P^{(k)}]$ i.e., $h_k = 0$ a.e. with respect to $P^{(k)}$, for all $P \varepsilon \Omega$, whenever h_k satisfies

(a) h_k is a symmetric function [measurable on $(X^{(k)}, \mathcal{S}^{(k)})$]; and

(b) $\int h_k \, dP^{(k)} = 0$ for all $P \varepsilon \Omega$.

With this notation we now demonstrate that the class $\Omega_2(X)$ is symmetrically complete for all n.

In the sequel it will be assumed that ν is an arbitrary fixed *nonatomic prob-*

ability measure on $(X, \mathcal{S})$; that h_n is a symmetric measurable function on $(X^{(n)}, \mathcal{S}^{(n)})$; and that $\mathcal{A}$ is a *semi-algebra* which generates $\mathcal{S}$. [Note: $\mathcal{A}$ is a semi-algebra if $X \varepsilon \mathcal{A}$; $\mathcal{A}$ is closed under finite intersections; and $A, B \varepsilon \mathcal{A}$ with $A \subset B$ implies the existence of $\{A_0, A_1, \cdots, A_m\} \subset \mathcal{A}$ such that $A = A_0 \subset A_1 \subset \cdots \subset A_m = B$ and $A_i - A_{i-1} \varepsilon \mathcal{A}$ for $i = 1, 2, \cdots, m$.]

3. The main theorem. The proof of the main theorem utilizes the facts that $\Omega(\mathcal{A}, \gamma)$ is symmetrically complete for properly chosen $\mathcal{A} \subset \mathcal{S}$; that the null classes of $\Omega^{(n)}(\mathcal{A}, P_1)$ and $\Omega_3^{(n)}(P_1)$ are equal; that, therefore, $\Omega_3(P_1)$ is symmetrically complete; and that so is $\Omega_2(X)$, since it is the union of classes $\Omega_3(P)$.

These ideas are given more precisely by the following three lemmas.

LEMMA 1. (Fraser) *If γ is an arbitrary nonatomic probability measure on $(X, \mathcal{S})$ and $\mathcal{A}$ is a semi-algebra which generates $\mathcal{S}$, then $\Omega(\mathcal{A}, \gamma)$ is symmetrically complete for all n.*

PROOF. See Fraser [2].

LEMMA 2. *If $P_1 \varepsilon \Omega_2(X)$, then $\mathfrak{N}_{\Omega^{(n)}(\mathcal{A}, P_1)} = \mathfrak{N}_{\Omega_3^{(n)}(P_1)}$ for all n.*

PROOF. Let n be an arbitrary fixed positive integer. Clearly, $P_1^{(n)}(A) = 0$ implies $P^{(n)}(A) = 0$ for all $P \varepsilon \Omega_3(P_1)$. This latter condition implies $\mathfrak{N}_{\{P_1^{(n)}\}} \subset \mathfrak{N}_{\Omega_3^{(n)}(P_1)}$. On the other hand, since

$$P_1^{(n)} \varepsilon \Omega^{(n)}(\mathcal{A}, P_1) \subset \Omega_3^{(n)}(P_1), \mathfrak{N}_{\{P_1^n\}} \supset \mathfrak{N}_{\Omega^{(n)}(\mathcal{A}, P_1)} \supset \mathfrak{N}_{\Omega_3^{(n)}(P_1)} .$$

The conclusion follows immediately.

The symmetric completeness of $\Omega(\mathcal{A}, P_1)$ and the equality of the two null classes are sufficient to establish the next lemma.

LEMMA 3. *If $P_1 \varepsilon \Omega_2(X)$, then $\Omega_3(P_1)$ is symmetrically complete for all n.*

PROOF. $\int h_n \, dP^{(n)} = 0$ for all $P \varepsilon \Omega_3(P_1)$ implies $P^{(n)}\{h_n \neq 0\} = 0$ for all $P \varepsilon \Omega(\mathcal{A}, P_1) \subset \Omega_3(P_1)$. Hence $\{h_n \neq 0\} \varepsilon \mathfrak{N}_{\Omega^{(n)}(\mathcal{A}, P_1)} = \mathfrak{N}_{\Omega_3^{(n)}(P_1)}$ and $h_n \equiv 0[P^{(n)}]$ for all $P \varepsilon \Omega_3(P_1)$.

The main theorem now follows from the preceding lemmas and the fact that any measure absolutely continuous with respect to a nonatomic measure is itself nonatomic.

MAIN THEOREM. *The class $\Omega_2(X)$ of all nonatomic probability measures on an arbitrary measurable space $(X, \mathcal{S})$ is a symmetrically complete class for all n. In particular, the class Ω_2 of all continuous cpf's on the real line is a symmetrically complete class for all n.*

PROOF. It is sufficient to demonstrate that for arbitrary fixed n, and arbitrary fixed $P_1 \varepsilon \Omega_2(X)$, $P_1^{(n)}\{h_n \neq 0\} = 0$, whenever h_n is a measurable symmetric function with the property: $\int h_n \, dP^{(n)} = 0$ for all $P \varepsilon \Omega_2(X)$.

Under such circumstances it is clear that $\Omega_3(P_1) \subset \Omega_2(X)$. Therefore, Lemma 3 guarantees for symmetric h_n such that $\int h_n \, dP^{(n)} = 0$ for all $P \varepsilon \Omega_2(X)$, that $P^{(n)}\{h_n \neq 0\} = 0$ for all $P \varepsilon \Omega_3(P_1)$. But $P_1 \varepsilon \Omega_3(P_1)$ and, consequently, $P_1^{(n)}\{h_n \neq 0\} = 0$.

4. Extensions. The symmetric completeness of several other classes of statistical interest can be extended to abstract spaces. In fact, by an extension of the ideas above and those of Fraser ([2],[3], pp. 23–31), one can demonstrate the following result.

THEOREM. *If* $(X, \mathcal{S})$ *is an arbitrary measurable space, then* (I) $\Omega_0(X)$, $\Omega_1(X)$ *and* $\Omega_2(X)$ *are symmetrically complete for all n.*

If, further, λ *is a nonatomic,* σ*-finite measure on* $\mathcal{S}$ *and* $\mathcal{A}$ *is a semialgebra which generates* $\mathcal{S}$*, then,* (II) $\Omega(\mathcal{A}, \lambda)$, $\Omega(\mathcal{S}, \lambda)$ *and* $\Omega_3(\lambda)$ *are symmetrically complete for all n.*

REFERENCES

[1] C. B. BELL, "On the structure of distribution-free statistics," *Ann. Math. Stat.*, Vol. 31 (1960), pp. 703–709.
[2] D. A. S. FRASER, "Completeness of order statistics," *Can. J. Math.*, Vol. 6 (1954), pp. 42–45.
[3] D. A. S. FRASER, *Non-Parametric Methods in Statistics*, John Wiley and Sons, New York, 1957.
[4] E. L. LEHMANN, *Testing Statistical Hypotheses*, John Wiley and Sons, New York, 1959.
[5] E. L. LEHMANN AND HENRY SCHEFFÉ, "Completeness, similar regions and unbiased estimation," Part I, *Sankhyā*, Vol. 10 (1950), pp. 305–340.
[6] E. L. LEHMANN AND HENRY SCHEFFÉ, "Completeness, similar regions and unbiased estimation," Part II, *Sankhyā*, Vol. 15 (1955), pp. 219–236.
[7] H. SCHEFFÉ, "On a measure problem arising in the theory of non-parametric tests," *Ann. Math. Stat.*, Vol. 14 (1943), pp. 227–233.

ON CENTERING INFINITELY DIVISIBLE PROCESSES[1]

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The concept of centering stochastic processes having independent increments, introduced by Lévy, is applied to processes having both stationary and independent increments. The main purpose of this note is to answer the question as to what centering functions preserve the stationarity of the increments.

In 1934, Lévy [1] proved that any stochastic process with independent increments may be transformed by subtraction of a sure function, called a centering function, into a process whose sample functions possess certain desirable smoothness properties. (cf. Lévy [2] and Doob [3]). It is clear that the transformed process, called the centered process, is also a process possessing independent increments. The purpose of this paper is to show that a process having stationary and independent increments may be centered in such a way so as to preserve the stationarity as well as the independence of the increments.

To be more precise, consider the following definitions (cf. Doob [3] p. 407).

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