多元统计分析第一次作业

学习交流，无限进步

2024 年 9 月 5 日

Exercise 1

7. 设 $X^{(1)}, X^{(2)}$ 是 p 维随机向量，且

$$X = \begin{pmatrix} X^{(1)} \\ X^{(2)} \end{pmatrix} \sim N_{2p}\left(\begin{pmatrix} \mu^{(1)} \\ \mu^{(2)} \end{pmatrix}, \begin{pmatrix} \Sigma_1 & \Sigma_2 \\ \Sigma_2 & \Sigma_1 \end{pmatrix} \right),$$

其中 $\mu^{(1)}$ 和 $\mu^{(2)}$ 为 p 维列向量，Σ_1 和 Σ_2 为 p 阶正定矩阵。
(1) 试证 $X^{(1)} + X^{(2)}$ 与 $X^{(1)} - X^{(2)}$ 相互独立;
(2) 试求 $X^{(1)} + X^{(2)}$ 与 $X^{(1)} - X^{(2)}$ 的分布。

证明. (1) 由于：

$$\begin{pmatrix} X^{(1)} + X^{(2)} \\ X^{(1)} - X^{(2)} \end{pmatrix} = \begin{pmatrix} I_p & -I_p \\ I_p & I_p \end{pmatrix} \begin{pmatrix} X^{(1)} \\ X^{(2)} \end{pmatrix}$$

由多元正态的线性性：

$$\begin{pmatrix} X^{(1)} + X^{(2)} \\ X^{(1)} - X^{(2)} \end{pmatrix} \sim N_{2p}\left(\begin{pmatrix} \mu^{(1)} + \mu^{(2)} \\ \mu^{(1)} - \mu^{(2)} \end{pmatrix}, \begin{pmatrix} I_p & -I_p \\ I_p & I_p \end{pmatrix} \begin{pmatrix} \Sigma_1 & \Sigma_2 \\ \Sigma_2 & \Sigma_1 \end{pmatrix} \begin{pmatrix} I_p & -I_p \\ I_p & I_p \end{pmatrix}' \right)$$

即：

$$\begin{pmatrix} X^{(1)} + X^{(2)} \\ X^{(1)} - X^{(2)} \end{pmatrix} \sim N_{2p}\left(\begin{pmatrix} \mu^{(1)} + \mu^{(2)} \\ \mu^{(1)} - \mu^{(2)} \end{pmatrix}, \begin{pmatrix} 2\Sigma_1 - 2\Sigma_2 & 0 \\ 0 & 2\Sigma_1 + 2\Sigma_2 \end{pmatrix} \right)$$

于是 $X^{(1)} + X^{(2)}$ 与 $X^{(1)} - X^{(2)}$ 相互独立.

(2) 由于：

$$\begin{pmatrix} X^{(1)}+X^{(2)} \\ X^{(1)}-X^{(2)} \end{pmatrix} \sim N_{2p}\left(\begin{pmatrix} \mu^{(1)}+\mu^{(2)} \\ \mu^{(1)}-\mu^{(2)} \end{pmatrix}, \begin{pmatrix} 2\Sigma_1-2\Sigma_2 & 0 \\ 0 & 2\Sigma_1+2\Sigma_2 \end{pmatrix}\right)$$

于是

$$X^{(1)}+X^{(2)} \sim N_p\left(\mu^{(1)}+\mu^{(2)}, 2\Sigma_1-2\Sigma_2\right)$$

$$X^{(1)}-X^{(2)} \sim N_p\left(\mu^{(1)}-\mu^{(2)}, 2\Sigma_1+2\Sigma_2\right)$$

□

Exercise 2

8. 设 $X \sim N_p(\mu, \Sigma)$, $\Sigma > 0$，对 μ 和 Σ 作如下剖分：

$$\mu = \begin{pmatrix} \mu^{(1)} \\ \mu^{(2)} \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix},$$

其中 $\mu^{(1)}$ 为 r 维列向量，Σ_{11} 为 r 阶方阵，$1 \leqslant r < p$。
(1) 试证明：$\mu'\Sigma^{-1}\mu \geqslant (\mu^{(1)})'\Sigma_{11}^{-1}\mu^{(1)}$；
(2) 试证明：$X^{(2)} \mid X^{(1)} = x^{(1)} \sim N_{p-r}\left(\mu_{2.1}, \Sigma_{22.1}\right)$，其中

$$\mu_{2.1} = \mu^{(2)} + \Sigma_{21}\Sigma_{11}^{-1}(x^{(1)} - \mu^{(1)}) \quad \text{和} \quad \Sigma_{22.1} = \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}.$$

证明. (1) 由分块矩阵求逆

$$\Sigma^{-1} = \begin{pmatrix} \Sigma_{11}^{-1} + \Sigma_{11}^{-1}\Sigma_{12}\Sigma_{22.1}^{-1}\Sigma_{21}\Sigma_{11}^{-1} & -\Sigma_{11}^{-1}\Sigma_{12}\Sigma_{22.1}^{-1} \\ -\Sigma_{22.1}^{-1}\Sigma_{21}\Sigma_{11}^{-1} & \Sigma_{22.1}^{-1} \end{pmatrix}$$

其中 $\Sigma_{22.1}=\Sigma_{22}-\Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}$ 于是

$$\begin{aligned}\mu'\Sigma^{-1}\mu=&(\mu^{(1)})'\Sigma_{11}^{-1}\mu^{(1)}+(\mu^{(1)})'\Sigma_{11}^{-1}\Sigma_{12}\Sigma_{22.1}^{-1}\Sigma_{21}\Sigma_{11}^{-1}\mu^{(1)}-(\mu^{(1)})'\Sigma_{11}^{-1}\Sigma_{12}\Sigma_{22.1}^{-1}\mu^{(2)}\\&-(\mu^{(2)})'\Sigma_{22.1}^{-1}\Sigma_{21}\Sigma_{11}^{-1}\mu^{(1)}+(\mu^{(2)})'\Sigma_{22.1}^{-1}\mu^{(2)}\\=&(\mu^{(1)})'\Sigma_{11}^{-1}\mu^{(1)}+(\mu^{(2)})'\Sigma_{22.1}^{-1}(\mu^{(2)}-\Sigma_{21}\Sigma_{11}^{-1}\mu^{(1)})\\&-(\mu^{(1)})'\Sigma_{11}^{-1}\Sigma_{12}\Sigma_{22.1}^{-1}(\mu^{(2)}-\Sigma_{21}\Sigma_{11}^{-1}\mu^{(1)})\\=&(\mu^{(1)})'\Sigma_{11}^{-1}\mu^{(1)}+(\mu^{(2)}-\Sigma_{21}\Sigma_{11}^{-1}\mu^{(1)})'\Sigma_{22.1}^{-1}(\mu^{(2)}-\Sigma_{21}\Sigma_{11}^{-1}\mu^{(1)})\end{aligned}$$

下证明 $\Sigma_{22.1}^{-1}$ 的半正定性:

$$\Sigma^{-1}=\begin{pmatrix} I_q & -\Sigma_{11}^{-1}\Sigma_{12}\\ 0 & I_{p-q}\end{pmatrix}\begin{pmatrix}\Sigma_{11}^{-1} & 0\\ 0 & \Sigma_{22.1}^{-1}\end{pmatrix}\begin{pmatrix} I_q & 0\\ -\Sigma_{21}\Sigma_{11}^{-1} & I_{p-q}\end{pmatrix}\quad *$$

记 $\Sigma^{-1}=P\begin{pmatrix}\Sigma_{11}^{-1} & 0\\ 0 & \Sigma_{22.1}^{-1}\end{pmatrix}P'$ ，其中 P 均为可逆矩阵
则 $\begin{pmatrix}\Sigma_{11}^{-1} & 0\\ 0 & \Sigma_{22.1}^{-1}\end{pmatrix}=P^{-1}\Sigma^{-1}(P')^{-1}$
于是任取向量 V ，记 $U=(P')^{-1}V$ ，由 Σ^{-1} 的正定性:

$$V'\begin{pmatrix}\Sigma_{11}^{-1} & 0\\ 0 & \Sigma_{22.1}^{-1}\end{pmatrix}V=V'P^{-1}\Sigma^{-1}(P')^{-1}V=U'\Sigma^{-1}U\geq 0$$

于是 $\begin{pmatrix}\Sigma_{11}^{-1} & 0\\ 0 & \Sigma_{22.1}^{-1}\end{pmatrix}$ 半正定，进而 $\Sigma_{22.1}^{-1}$ 半正定.
因此 $(\mu^{(2)}-\Sigma_{21}\Sigma_{11}^{-1}\mu^{(1)})'\Sigma_{22.1}^{-1}(\mu^{(2)}-\Sigma_{21}\Sigma_{11}^{-1}\mu^{(1)})\geq 0$ ，于是 $\mu'\Sigma^{-1}\mu\geq(\mu^{(1)})'\Sigma_{11}^{-1}\mu^{(1)}$
(2) 由多元正态分布密度函数定义:

$$f(X)=\frac{1}{(2\pi)^{\frac{p}{2}}|\Sigma|^{\frac{1}{2}}}exp\{-\frac{1}{2}(X-\mu)'\Sigma^{-1}(X-\mu)\}=f(X^{(2)}\mid X^{(1)})f(X^{(1)})$$

由于 $X^{(1)}\sim N_p(\mu^{(1)},\Sigma_{11})$，于是

$$f(X^{(1)})=\frac{1}{(2\pi)^{\frac{r}{2}}|\Sigma_{11}|^{\frac{1}{2}}}exp\{-\frac{1}{2}(X^{(1)}-\mu^{(1)})'\Sigma_{11}^{-1}(X^{(1)}-\mu^{(1)})\}$$

由 * 知 $|\Sigma^{-1}|=|\Sigma_{11}^{-1}||\Sigma_{22.1}^{-1}|$
计算可得:

$$
\begin{aligned}
f(X^{(2)} \mid X^{(1)}) =& \frac{|\Sigma_{11}|^{\frac{1}{2}}}{(2\pi)^{\frac{p-r}{2}}|\Sigma|^{-\frac{1}{2}}} exp\{\frac{1}{2}(X^{(1)}-\mu^{(1)})'\Sigma_{11}^{-1}(X^{(1)}-\mu^{(1)}) - \frac{1}{2}(X-\mu)'\Sigma^{-1}(X-\mu)\} \\
=& \frac{|\Sigma_{11}|^{\frac{1}{2}}}{(2\pi)^{\frac{p-r}{2}}|\Sigma|^{-\frac{1}{2}}} exp\{-\frac{1}{2}((X^{(2)}-\mu^{(2)}) - \Sigma_{21}\Sigma_{11}^{-1}(X^{(1)}-\mu^{(1)})))' \\
& \Sigma_{22.1}^{-1}((X^{(2)}-\mu^{(2)}) - \Sigma_{21}\Sigma_{11}^{-1}(X^{(1)}-\mu^{(1)}))\} \\
=& \frac{1}{(2\pi)^{\frac{p-r}{2}}|\Sigma_{22.1}|^{-\frac{1}{2}}} exp\{-\frac{1}{2}((X^{(2)} - (\mu^{(2)} + \Sigma_{21}\Sigma_{11}^{-1}(x^{(1)}-\mu^{(1)}))))' \\
& \Sigma_{22.1}^{-1}(X^{(2)} - (\mu^{(2)} + \Sigma_{21}\Sigma_{11}^{-1}(x^{(1)}-\mu^{(1)}))))\}
\end{aligned}
$$

于是 $X^{(2)} \mid X^{(1)} = x^{(1)} \sim N_{p-r}\left(\mu_{2.1}, \Sigma_{22.1}\right)$ □

Exercise 3

14. 令 $X_1, \cdots, X_n$ 是相互独立的，且 $X_i \sim N\left(\beta + \gamma z_i, \sigma^2\right)$，其中 z_i 是给定的常数，$i = 1, \cdots, n$，且 $\sum_{i=1}^{n} z_i = 0$。
(1) 求 $(X_1, \cdots, X_n)'$ 的分布；
(2) 求 $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ 和 $Y = \frac{\sum_{i=1}^{n} z_i X_i}{\sum_{i=1}^{n} z_i^2}$ 的分布，其中 $\sum_{i=1}^{n} z_i^2 > 0$。

证明. (1) 由独立性：

$$
\begin{aligned}
F((x_1, \cdots, x_n)') =& P(X_1 \leq x_1, \cdots, X_n \leq x_n) \\
=& \prod_{i=1}^{n} P(X_i \leq x_i) \\
=& \prod_{i=1}^{n} F_{X_i}(x_i)
\end{aligned}
$$

于是：

$$
\begin{aligned}
f((x_1,\cdots,x_n)') =& \prod_{i=1}^{n} f_{X_i}(x_i) \\
=& \prod_{i=1}^{n} \frac{1}{(2\pi)^{\frac{1}{2}}\sigma} exp\{-\frac{1}{2\sigma^2}(x_i-(\beta+\gamma z_i)^2\} \\
=& \frac{1}{(2\pi)^{\frac{n}{2}}|\Sigma|^{\frac{1}{2}}} exp\{-\frac{1}{2}(X-\mu)'\Sigma^{-1}(X-\mu)\}
\end{aligned}
$$

$$= f(X^{(2)} \mid X^{(1)})f(X^{(1)})$$

其中 $X=(x_1,\cdots,x_n)'$ ， $\Sigma=\sigma^2 I_n$ ， $\mu=(\beta+\gamma z_1,\cdots,\beta+\gamma z_n)'$

即 $X \sim N_p(\mu,\Sigma)$。

(2) 由于：

$$
\begin{aligned}
\bar{X} =& \frac{1}{n}\sum_{i=1}^{n} X_i = \frac{1}{n}1_n'X \\
Y =& \frac{\sum_{i=1}^{n} z_i X_i}{\sum_{i=1}^{n} z_i^2} \\
=& \frac{1}{\sum_{i=1}^{n} z_i^2}(z_1,\cdots,z_n)X
\end{aligned}
$$

记 $Z=\frac{1}{\sum_{i=1}^{n} z_i^2}(z_1,\cdots,z_n)$

于是：

$$
\begin{aligned}
\bar{X} \sim N(\frac{1}{n}1_n'\mu, \frac{1}{n^2}1_n'\Sigma 1_n) \\
Y \sim N(Z\mu, Z\Sigma Z')
\end{aligned}
$$

即：

$$
\begin{aligned}
\bar{X} \sim N(\beta, \frac{\sigma^2}{n}) \\
Y \sim N(\gamma, \sigma^2)
\end{aligned}
$$

□

Exercise 4

19. 令 $a = \begin{pmatrix} -4 \\ 3 \end{pmatrix}$ 和 $b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$，以及

$$A = \begin{pmatrix} 2 & -2 \\ -2 & 5 \end{pmatrix},$$

试证明推广的 Cauchy-Schwarz 不等式：

$$(a'b)^2 \leqslant (a'Aa)(b'A^{-1}b).$$

证明.

$$\begin{aligned} a'b &= -1 \\ a'Aa &= 125 \\ A^{-1} &= \begin{pmatrix} \frac{5}{6} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix} \\ b'A^{-1}b &= \frac{5}{6} \end{aligned}$$

显然

$$(a'b)^2 \leqslant (a'Aa)(b'A^{-1}b).$$

□

Exercise 5

21. 试证明

$$\begin{aligned} \Sigma^{-1} &= \begin{pmatrix} \Sigma_{11.2}^{-1} & -\Sigma_{11.2}^{-1}\Sigma_{12}\Sigma_{22}^{-1} \\ -\Sigma_{22}^{-1}\Sigma_{21}\Sigma_{11.2}^{-1} & \Sigma_{22}^{-1}\Sigma_{21}\Sigma_{11.2}^{-1}\Sigma_{12}\Sigma_{22}^{-1} + \Sigma_{22}^{-1} \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & \Sigma_{22}^{-1} \end{pmatrix} + \begin{pmatrix} I \\ \beta' \end{pmatrix} \Sigma_{11.2}^{-1}(I - \beta), \end{aligned}$$

其中 $\beta=\Sigma_{12}\Sigma_{22}^{-1}$.

证明.

$$
\begin{pmatrix}\Sigma_{11} & \bar{\Sigma}_{12}\\ \Sigma_{21} & \bar{\Sigma}_{22}\end{pmatrix}=\begin{pmatrix}1 & \Sigma_{12}\Sigma_{22}^{-1}\\ 0 & I\end{pmatrix}\begin{pmatrix}\bar{\Sigma}_{11}-\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} & 0\\ \bar{\Sigma}_{11} & \bar{\Sigma}_{22}\end{pmatrix}
$$

$$
=\begin{pmatrix}I & \Sigma_{12}\Sigma_{22}^{-1}\\ 0 & I\end{pmatrix}\begin{pmatrix}\Sigma_{11}-\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} & 0\\ 0 & \Sigma_{21}\end{pmatrix}\begin{pmatrix}I & 0\\ \Sigma_{22}\Sigma_{21} & I\end{pmatrix}
$$

于是:

$$
\begin{aligned}
\Sigma^{-1}&=\begin{pmatrix}I & 0\\ -\Sigma_{22}^{-1}\Sigma_{21} & I\end{pmatrix}\begin{pmatrix}\Sigma_{11.2}^{-1} & 0\\ 0 & \Sigma_{22}^{-1}\end{pmatrix}\begin{pmatrix}I & -\Sigma_{12}\Sigma_{22}^{-1}\\ 0 & I\end{pmatrix}\\
&=\begin{pmatrix}\Sigma_{11.2}^{-1} & 0\\ -\Sigma_{22}^{-1}\Sigma_{21}\Sigma_{11.2}^{-1} & \Sigma_{22}^{-1}\end{pmatrix}\begin{pmatrix}I & -\Sigma_{12}\Sigma_{22}^{-1}\\ 0 & I\end{pmatrix}\\
&=\begin{pmatrix}\Sigma_{11.2}^{-1} & -\Sigma_{11.2}^{-1}\Sigma_{12}\Sigma_{22}^{-1}\\ -\Sigma_{22}^{-1}\Sigma_{21}\Sigma_{11.2}^{-1} & \Sigma_{22}^{-1}\end{pmatrix}\\
&=\begin{pmatrix}0 & 0\\ 0 & \Sigma_{22}^{-1}\end{pmatrix}+\begin{pmatrix}\Sigma_{11.2}^{-1} & -\Sigma_{11.2}^{-1}\Sigma_{12}\Sigma_{22}^{-1}\\ -\Sigma_{22}^{-1}\Sigma_{21}\Sigma_{11.2}^{-1} & 0\end{pmatrix}\\
&=\begin{pmatrix}0 & 0\\ 0 & \Sigma_{22}^{-1}\end{pmatrix}+\begin{pmatrix}I\\ \beta'\end{pmatrix}\Sigma_{11.2}^{-1}(I-\beta)
\end{aligned}
$$

其中 $\beta=\Sigma_{12}\Sigma_{22}^{-1}$ ， $\Sigma_{11.2}=\Sigma_{11}-\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$ □